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NUMERICAL DIFFERENTIATION USING ORTHOGONAL POLYNOMIALS

by

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U. S. ARMY MATERIEL COMMAND
BALLISTIC RESEARCH LABORATORIES
ABERDEEN PROVING GROUND, MARYLAND

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ABSTRACT

The present method of reducing data by formulating a polynomial, taking the derivative and substituting experimental values, has been simplified for instances where the independent variable is evenly spaced and the derivative is desired at these given points only.

Orthogonal polynomials, resembling Legendre polynomials, are differentiated and tabulated in a useful form. When used in conjunction with available orthogonal polynomial tables, these new tables give derivatives more readily than the usual procedure. These tables present coefficients for obtaining first derivatives at 6 to 21 evenly spaced points. This report extends the available orthogonal polynomial tables to sixth and seventh degrees, when appropriate, and gives a demonstration of the method.

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INTRODUCTION

Numerical differentiation calculates approximate values of derivatives from given discrete values of the function itself. The functional values determine an interpolating function whose derivative approximates derivatives of the original function. Commonly, the interpolation function is a polynomial which fits the given points exactly, or it is a polynomial of lower degree determined by the method of least squares.

Numerical data obtained from experiments require polynomials of lower degree, since fitting given points exactly would lead to derivatives which are too sensitive to experimental errors in the original data. If the values of the independent variable are uniformly spaced, the method of orthogonal polynomials greatly simplifies the calculation. Tables of coefficients for using this method appear in Milne's Numerical Calculus.* To use this method, the polynomial must be formulated and differentiated.

The study reported here avoids this laborious process when the quantities of interest occur at the given data points. Tables have been constructed which give the values of the derivative at the uniformly spaced points, avoiding the explicit conversion to the original independent variable and combining like powers; these tables are found in the Appendix. The successive contributions of the orthogonal polynomials of various degrees are calculated separately, thus providing a simple criterion for selecting the best degree to use. The degree must be high enough to follow the significant variation of the data but not so high as to destroy the particular direction of the original data.

The following sections present the approximating polynomial and the derivative, and they describe how the new tables are constructed for the derivative and how they are used.

*William Edmund Milne, Numerical Calculus, 2nd ed., Princeton, New Jersey, Princeton University Press, 1949, Chapter 9, pp. 265-275.

FORMULATION

Let (x_0,y_0), (x_1,y_1), . . . , (x_n,y_n) denote $(n + 1)$ corresponding pairs of values of any two variables x and y. If the values of x are free of error and equidistant (interval h), the independent variable x can be represented by the points, 0, 1, . . . , n in the transformation $s = (x - x_0)/h$ and $y(x) = f(s)$. The approximating polynomial of the mth degree can be written*

$$Q_m(s) = \sum_{k=0}^{m} \left[\frac{\sum_{s=0}^{n} P_{k,n}(s) f(s)}{\sum_{s=0}^{n} P_{k,n}^2(s)} \right] P_{k,n}(s) \equiv \sum_{k=0}^{m} \frac{c_k}{S_k} P_{k,n}(s)$$

where**

$$P_{k,n}(s) = \sum_{r=0}^{k} (-1)^r \binom{k}{r} \binom{k+r}{r} \frac{s^{(r)}}{n^{(r)}} .$$

The fractions c_k/S_k form a unique set of constants which yield the most probable equation through the least square technique. With the functional values from the data and available tables of orthogonal polynomial values,* these constants can readily be obtained.

Differentiating this approximating polynomial with respect to x, we have

$$Q_m'(x) = \frac{1}{h} Q_m'(s) = \frac{1}{h} \sum_{k=0}^{m} \frac{c_k}{S_k} P_{k,n}'(s) .$$

Since the quantities $P_{k,n}'(s)$ do not depend on $f(s)$ they can be computed once and for all and presented in tabular form.

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*Milne

** $s^{(r)} = s(s - 1)(s - 2) \cdots (s - r + 1)$

$\binom{k}{r} = \frac{k!}{r!\,(k - r)!}$.

CONSTRUCTION AND USE OF THE TABLES

The tables in the Appendix of this report are constructed from a useful form of the derivatives. The entries in each column of the available orthogonal polynomial tables are numerators of numbers with their common denominator being the uppermost entry of its column.* Since the approximating polynomials take into account the denominators, the derivative tables had to be modified so that they could be used in conjunction with the available tables. The tabulated expressions $P'_{k,n}(s)$ are multiplied by the corresponding $s = 0$ value of the existing orthogonal polynomial tables. These products again are fractions, and since integral coefficients are desired in the derivative tables, a common denominator is determined for each column. These in turn are multiplied by h to take care of the change of the variable. These denominators, D, are indicated at the base of their respective columns in the derivative tables. It is also common practice to express $P_{k,n}(x)$ and $P'_{k,n}(x)$ in the tables as $P_k(x)$ and $P'_k(x)$ since n is indicated in the top line of each table. Also $P_0(x)$ is not shown since its entries are always unity.

To illustrate the use of the tables, consider the simple function $y = e^x$ where x takes on values 0 to 0.9 and the interval $h = 0.1$. Assume the data can be fitted with a fifth degree polynominal. It is well known that the derivation of $y = e^x$ is e^x. The appropriate orthogonal polynomial table and initial calculations can be seen in Example 1, Table 1a. The coefficients in row (C) are the quotients of (B) divided by (A). (A) is the sum of squares of the entries in the column above. (B) is the sum of the products of the entries in this column multiplied by their corresponding y's. For example, the coefficient in row (C) at the base of the column $P_1(s)$ is

$$\left[\frac{(1.00000)(9) + (1.10517)(7) + \cdot\cdot\cdot + (2.22554)(-7)}{330} + \frac{(2.45960)(-9)}{330}\right] = -0.0803467 .$$

*Milne

EXAMPLE 1

(Table 1a) Orthogonal Polynomial Table for n = 9(10 points), (See Appendix)

x	s	y		$P_0(s)$	$P_1(s)$	$P_2(s)$	$P_3(s)$	$P_4(s)$	$P_5(s)$
0	0	1.00000		1	9	6	42	18	6
0.1	1	1.10517		1	7	2	-14	-22	-14
0.2	2	1.22140		1	5	-1	-35	-17	1
0.3	3	1.34986		1	3	-3	-31	3	11
0.4	4	1.49182		1	1	-4	-12	18	6
0.5	5	1.64872		1	-1	-4	12	18	-6
0.6	6	1.82212		1	-3	-3	31	3	-11
0.7	7	2.01375		1	-5	-1	35	-17	-1
0.8	8	2.22554		1	-7	2	14	-22	14
0.9	9	2.45960		1	-9	6	-42	18	-6
			(A)	10	330	132	8580	2820	780
			(B)	16.33798	-26.51442	2.10577	-1.36291	0.04529	-0.00103
			(C)	1.633798	-0.0803467	0.015953	-0.0001588	0.0000161	-0.0000013

(Table 1b) Numerical Differentiation Table for n = 9(10 points), (See Appendix)

s		$P_1'(s)$	$P_2'(s)$	$P_3'(s)$	$P_4'(s)$	$P_5'(s)$
0		-2	-9	-461	-225	-911
1		-2	-7	-221	-35	109
2		-2	-5	-41	50	244
3		-2	-3	79	60	34
4		-2	-1	139	25	-161
5		-2	1	139	-25	-161
6		-2	3	79	-60	34
7		-2	5	-41	-50	244
8		-2	7	-221	35	109
9		-2	9	-461	225	-911
	(D)	1h	2h	6h	3h	15h

(Table 1c) Individual Contributions of the Various Degrees

s		$P_1'(s)$	$P_2'(s)$	$P_3'(s)$	$P_4'(s)$	$P_5'(s)$
0		1.606934	-0.717885	0.122165	-0.012150	0.000911
1		1.606934	-0.558355	0.058565	-0.001890	-0.000109
2		1.606934	-0.398825	0.010865	0.002700	-0.000244
3		1.606934	-0.239295	-0.020935	0.003240	-0.000034
4		1.606934	-0.079765	-0.030835	0.001350	0.000161
5		1.606934	0.079765	-0.030835	-0.001350	0.000161
6		1.606934	0.239295	-0.020935	-0.003240	-0.000034
7		1.606934	0.398825	0.010865	-0.002700	-0.000244
8		1.606934	0.558355	0.058565	0.001890	-0.000109
9		1.606934	0.717885	0.122165	0.012150	0.000911
	(D)	0.1	0.2	0.6	0.3	1.5
	(C/D)	-0.803467	0.079765	-0.000265	0.000054	-0.000001

EXAMPLE 1 (Cont'd)

(Table 1d) Cumulative Value of the Derivation For The Various Degrees

s	$P_1'(s)$	$P_2'(s)$	$P_3'(s)$	$P_4'(s)$	$P_5'(s)$
0	1.606934	0.889049	1.011214	0.999064	0.999975
1	1.606934	1.048579	1.107144	1.105254	1.105145
2	1.606934	1.208109	1.218974	1.221674	1.221430
3	1.606934	1.367639	1.346704	1.349944	1.349910
4	1.606934	1.527169	1.490334	1.491684	1.491845
5	1.606934	1.686699	1.649864	1.648514	1.648675
6	1.606934	1.846229	1.825294	1.822054	1.822020
7	1.606934	2.005759	2.016624	2.013924	2.013680
8	1.606934	2.165289	2.223854	2.225744	2.225635
9	1.606934	2.324819	2.446984	2.459134	2.460045

For the derivative calculations, the coefficients in row (C) of the orthogonal polynomial tables are divided by the denominators of the respective columns in the derivative tables [row (D) in Table 1b]. The newly computed values [row (C) in Table 1a divided by row (D) in Table 1b] are presented in row (C/D) of Table 1c. Each of the tabulated values in the columns of Table 1c is obtained by multiplying the value in row (C/D) by the individual entries in the corresponding column of Table 1b. The products are entered in rows 0-9 in Table 1c. These are the individual contributions. Table 1d displays the cumulative values. For instance, the value in Table 1d corresponding to the entry in row s = 4, under P_5' is

$$\frac{(-0.0803467)(-2)}{0.1} + \frac{(0.015953)(-1)}{0.2} + \frac{(-0.0001588)(139)}{0.6}$$

$$+ \frac{(0.0000161)(25)}{0.3} + \frac{(-0.0000013)(-161)}{1.5} = 1.491845 \ .$$

The first three terms give the value for the entry in the same row, s = 4, under P_3'. These values are indeed approximately the original values of $y = e^x$.

This example has shown the mechanics for using the tables to determine derivatives at the given points. Of course no local irregularities were noted in the fitting since the data was that of a smooth function and not experimentally obtained data.

SUMMARY

In the expectation that for many purposes derivatives at values of the independent variable in the original data would suffice, tables were constructed to facilitate this end. It is required that the data be evenly spaced. For convenience, extension of the orthogonal polynomial tables as in Milne were constructed to take into account fits of higher degrees.

ACKNOWLEDGMENT

We wish to express our thanks to Dr. R. F. Jackson of the University of Delaware for his helpful suggestions.

APPENDIX

Tables. Orthogonal Polynomials and Numerical Differentiation Tables for n + 1 Equally Spaced Points

$n = 5$ (6 points)

s	$P_1(s)$	$P_2(s)$	$P_3(s)$	$P_4(s)$	$P_5(s)$
0	5	5	5	1	1
1	3	-1	-7	-3	-5
2	1	-4	-4	2	10
3	-1	-4	4	2	-10
4	-3	-1	7	-3	5
5	-5	5	-5	1	-1
A	70	84	180	28	252

s	$P_1'(s)$	$P_2'(s)$	$P_3'(s)$	$P_4'(s)$	$P_5'(s)$
0	-2	-15	-137	-50	-1756
1	-2	-9	-17	12	554
2	-2	-3	43	11	-181
3	-2	3	43	-11	-181
4	-2	9	-17	-12	554
5	-2	15	-137	50	-1756
D	1h	2h	6h	3h	15h

Tables. Orthogonal Polynomials and Numerical Differentiation Tables for n + 1 Equally Spaced Points

n = 6(7 points)

s	$P_1(s)$	$P_2(s)$	$P_3(s)$	$P_4(s)$	$P_5(s)$	$P_6(s)$
0	3	5	1	3	1	1
1	2	0	-1	-7	-4	-6
2	1	-3	-1	1	5	15
3	0	-4	0	6	0	-20
4	-1	-3	1	1	-5	15
5	-2	0	1	-7	4	-6
6	-3	5	-1	3	-1	1
A	28	84	6	154	84	924

s	$P_1'(s)$	$P_2'(s)$	$P_3'(s)$	$P_4'(s)$	$P_5'(s)$	$P_6'(s)$
0	-1	-6	-20	-177	-1207	-3479
1	-1	-4	-5	22	368	924
2	-1	-2	4	53	53	-413
3	-1	0	7	[illegible]	-262	0
4	-1	2	4	-53	53	413
5	-1	4	-5	-22	368	-924
6	-1	6	-20	177	-1207	3479
D	1h	1h	6h	6h	3[illegible]h	1[illegible]h

Tables. Orthogonal Polynomials and Numerical Differentiation Tables for n + 1 Equally Spaced Points

n = 7(8 points)

s	$P_1(s)$	$P_2(s)$	$P_3(s)$	$P_4(s)$	$P_5(s)$	$P_6(s)$	$P_7(s)$
0	7	7	7	7	7	1	1
1	5	1	-5	-13	-23	-5	-7
2	3	-3	-7	-3	17	9	21
3	1	-5	-3	9	15	-5	-35
4	-1	-5	3	9	-15	-5	35
5	-3	-3	7	-3	-17	9	-21
6	-5	1	5	-13	23	-5	7
7	-7	7	-7	7	-7	1	-1
A	168	168	264	616	2184	264	3432

s	$P_1'(s)$	$P_2'(s)$	$P_3'(s)$	$P_4'(s)$	$P_5'(s)$	$P_6'(s)$	$P_7'(s)$
0	-2	-7	-55	-287	-4519	-1029	-75823
1	-2	-5	-19	5	1151	305	16521
2	-2	-3	5	87	731	-61	-7643
3	-2	-1	17	43	-739	-127	2605
4	-2	1	17	-43	-739	127	2605
5	-2	3	5	-87	731	61	-7643
6	-2	5	-19	-5	1151	-305	16521
7	-2	7	-55	287	-4519	1029	-75823
D	1h	1h	3h	6h	30h	10h	70h

Tables. Orthogonal Polynomials and Numerical Differentiation Tables for n + 1 Equally Spaced Points

n = 8(9 points)

s	$P_1(s)$	$P_2(s)$	$P_3(s)$	$P_4(s)$	$P_5(s)$	$P_6(s)$	$P_7(s)$
0	4	28	14	14	4	4	1
1	3	7	-7	-21	-11	-17	-6
2	2	-8	-13	-11	4	22	14
3	1	-17	-9	9	9	1	-14
4	0	-20	0	18	0	-20	0
5	-1	-17	9	9	-9	1	14
6	-2	-8	13	-11	-4	22	-14
7	-3	7	7	-21	11	-17	6
8	-4	28	-14	14	-4	4	-1
A	60	2772	990	2002	468	1980	858

s	$P'_1(s)$	$P'_2(s)$	$P'_3(s)$	$P'_4(s)$	$P'_5(s)$	$P'_6(s)$	$P'_7(s)$
0	-1	-24	-181	-436	-1678	-1976	-39051
1	-1	-18	-76	-33	317	583	10096
2	-1	-12	-1	118	392	52	-3694
3	-1	-6	44	101	-103	-339	-1496
4	-1	0	59	0	-358	0	3985
5	-1	6	44	-101	-103	339	-1496
6	-1	12	-1	-118	392	-52	-3694
7	-1	18	-76	33	317	-583	10096
8	-1	24	-181	436	-1678	1976	-39051
D	1h	1h	6h	6h	30h	10h	140h

Tables. Orthogonal Polynomials and Numerical Differentiation Tables for n + 1 Equally Spaced Points

n = 9(10 points)

s	$P_1(s)$	$P_2(s)$	$P_3(s)$	$P_4(s)$	$P_5(s)$	$P_6(s)$	$P_7(s)$
0	9	6	42	18	6	3	9
1	7	2	-14	-22	-14	-11	-47
2	5	-1	-35	-17	1	10	86
3	3	-3	-31	3	11	6	-42
4	1	-4	-12	18	6	-8	-56
5	-1	-4	12	18	-6	-8	56
6	-3	-3	31	3	-11	6	42
7	-5	-1	35	-17	-1	10	-86
8	-7	2	14	-22	14	-11	47
9	-9	6	-42	18	-6	3	-9
A	330	132	8580	2860	780	660	29172

s	$P_1'(s)$	$P_2'(s)$	$P_3'(s)$	$P_4'(s)$	$P_5'(s)$	$P_6'(s)$	$P_7'(s)$
0	-2	-9	-461	-225	-911	-1779	-151399
1	-2	-7	-221	-35	109	483	42697
2	-2	-5	-41	50	244	190	-8466
3	-2	-3	79	60	34	-278	-16138
4	-2	-1	139	25	-161	-191	11981
5	-2	1	139	-25	-161	191	11981
6	-2	3	79	-60	34	278	-16138
7	-2	5	-41	-50	244	-190	-8466
8	-2	7	-221	35	109	-483	42697
9	-2	9	-461	225	-911	1779	-151399
D	1h	2h	6h	3h	15h	20h	140h

Tables. Orthogonal Polynomials and Numerical Differentiation Tables for n + 1 Equally Spaced Points

n = 10(11 points)

s	$P_1(s)$	$P_2(s)$	$P_3(s)$	$P_4(s)$	$P_5(s)$	$P_6(s)$	$P_7(s)$
0	5	15	30	6	3	15	5
1	4	6	-6	-6	-6	-48	-23
2	3	-1	-22	-6	-1	29	33
3	2	-6	-23	-1	4	36	-2
4	1	-9	-14	4	4	-12	-28
5	0	-10	0	6	0	-40	0
6	-1	-9	14	4	-4	-12	28
7	-2	-6	23	-1	-4	36	2
8	-3	-1	22	-6	1	29	-33
9	-4	6	6	-6	6	-48	23
10	-5	15	-30	6	-3	15	-5
A	110	858	4290	286	156	11220	4862

s	$P'_1(s)$	$P'_2(s)$	$P'_3(s)$	$P'_4(s)$	$P'_5(s)$	$P'_6(s)$	$P'_7(s)$
0	-1	-10	-286	-125	-1411	-6765	-139095
1	-1	-8	-151	-28	74	1424	40119
2	-1	-6	-46	21	389	1033	-159
3	-1	-4	29	34	164	-648	-20074
4	-1	-2	74	23	-151	-989	1549
5	-1	0	89	0	-286	0	17355
6	-1	2	74	-23	-151	989	1549
7	-1	4	29	-34	164	648	-20074
8	-1	6	-46	-21	389	-1033	-159
9	-1	8	-151	28	74	-1424	40119
10	-1	10	-286	125	-1411	6765	-139095
D	1h	1h	6h	6h	6[illegible]h	2[illegible]h	420h

Tables. Orthogonal Polynomials and Numerical Differentiation Tables for n + 1 Equally Spaced Points

n = 11(12 points)

s	$P_1(s)$	$P_2(s)$	$P_3(s)$	$P_4(s)$	$P_5(s)$	$P_6(s)$	$P_7(s)$
0	11	55	33	33	33	11	55
1	9	25	-3	-27	-57	-31	-225
2	7	1	-21	-33	-21	11	251
3	5	-17	-25	-13	29	25	83
4	3	-29	-19	12	44	4	-204
5	1	-35	-7	28	20	-20	-140
6	-1	-35	7	28	-20	-20	140
7	-3	-29	19	12	-44	4	204
8	-5	-17	25	-13	-29	25	-83
9	-7	1	21	-33	21	11	-251
10	-9	25	3	-27	57	-31	225
11	-11	55	-33	33	-33	11	-55
A	572	12012	5148	8008	15912	4488	369512

s	$P'_1(s)$	$P'_2(s)$	$P'_3(s)$	$P'_4(s)$	$P'_5(s)$	$P'_6(s)$	$P'_7(s)$
0	-2	-33	-139	-1177	-12581	-9967	-975225
1	-2	-27	-79	-333	-131	1987	273855
2	-2	-21	-31	133	3349	2121	55287
3	-2	-15	5	305	2179	-445	-146369
4	-2	-9	29	267	-401	-1711	-58533
5	-2	-3	41	103	-2231	-877	100115
6	-2	3	41	-103	-2231	877	100115
7	-2	9	29	-267	-401	1711	-58533
8	-2	15	5	-305	2179	445	-146369
9	-2	21	-31	-133	3349	-2121	55287
10	-2	27	-79	333	-131	-1987	273855
11	-2	33	-139	1177	-12581	9967	-975225
D	1h	1h	3h	12h	60h	60h	420h

Tables. Orthogonal Polynomials and Numerical Differentiation Tables for n + 1 Equally Spaced Points

$n = 12$ (13 points)

s	$P_1(s)$	$P_2(s)$	$P_3(s)$	$P_4(s)$	$P_5(s)$	$P_6(s)$	$P_7(s)$
0	6	22	11	99	22	22	33
1	5	11	0	-66	-33	-55	-121
2	4	2	-6	-96	-18	8	103
3	3	-5	-8	-54	11	43	75
4	2	-10	-7	11	26	22	-65
5	1	-13	-4	64	20	-20	-100
6	0	-14	0	84	0	-40	0
7	-1	-13	4	64	-20	-20	100
8	-2	-10	7	11	-26	22	65
9	-3	-5	8	-54	-11	43	-75
10	-4	2	6	-96	18	8	-103
11	-5	11	0	-66	33	-55	121
12	-6	22	-11	99	-22	22	-33
A	182	2002	572	68068	6188	14212	92378

s	$P_1'(s)$	$P_2'(s)$	$P_3'(s)$	$P_4'(s)$	$P_5'(s)$	$P_6'(s)$	$P_7'(s)$
0	-1	-12	-83	-1542	-7024	-15424	-409932
1	-1	-10	-50	-515	-479	2235	107739
2	-1	-8	-23	92	1726	3744	45033
3	-1	-6	-2	363	1481	253	-54045
4	-1	-4	13	382	256	-2368	-46660
5	-1	-2	22	233	-899	-2209	19063
6	-1	0	25	0	-1354	0	54189
7	-1	2	22	-233	-899	2209	19063
8	-1	4	13	-382	256	2368	-46660
9	-1	6	-2	-363	1481	-253	-54045
10	-1	8	-23	-92	1726	-3744	45033
11	-1	10	-50	515	-479	-2235	107739
12	-1	12	-83	1542	-7024	15424	-409932
D	1h	1h	6h	6h	60h	60h	420h

Tables. Orthogonal Polynomials and Numerical Differentiation Tables for n + 1 Equally Spaced Points

n = 13(14 points)

s	$P_1(s)$	$P_2(s)$	$P_3(s)$	$P_4(s)$	$P_5(s)$	$P_6(s)$	$P_7(s)$
0	13	13	143	143	143	143	143
1	11	7	11	-77	-187	-319	-473
2	9	2	-66	-132	-132	-11	297
3	7	-2	-98	-92	28	227	353
4	5	-5	-95	-13	139	185	-95
5	3	-7	-67	63	145	-25	-375
6	1	-8	-24	108	60	-200	-200
7	-1	-8	24	108	-60	-200	200
8	-3	-7	67	63	-145	-25	375
9	-5	-5	95	-13	-139	185	95
10	-7	-2	98	-92	-28	227	-353
11	-9	2	66	-132	132	-11	-297
12	-11	7	-11	-77	187	-319	473
13	-13	13	-143	143	-143	143	-143
A	910	728	97240	136136	235144	497420	1293292

s	$P'_1(s)$	$P'_2(s)$	$P'_3(s)$	$P'_4(s)$	$P'_5(s)$	$P'_6(s)$	$P'_7(s)$
0	-2	-13	-977	-988	-9794	-81536	-1328792
1	-2	-11	-617	-374	-1184	8008	315872
2	-2	-9	-317	9	2141	20937	205587
3	-2	-7	-77	203	2281	5831	-131197
4	-2	-5	103	250	916	-9520	-192440
5	-2	-3	223	192	-694	-13496	-22032
6	-2	-1	283	71	-1709	-6027	150567
7	-2	1	283	-71	-1709	6027	150567
8	-2	3	223	-192	-694	13496	-22032
9	-2	5	103	-250	916	9520	-192440
10	-2	7	-77	-203	2281	-5831	-131197
11	-2	9	-317	-9	2141	-20937	205587
12	-2	11	-617	374	-1184	-8008	315872
13	-2	13	-977	988	-9794	81536	-1328792
D	1h	2h	6h	3h	15h	60h	420h

Tables. Orthogonal Polynomials and Numerical Differentiation Tables for n + 1 Equally Spaced Points

n = 14(15 points)

s	$P_1(s)$	$P_2(s)$	$P_3(s)$	$P_4(s)$	$P_5(s)$	$P_6(s)$	$P_7(s)$
0	7	91	91	1001	1001	143	13
1	6	52	13	-429	-1144	-286	-39
2	5	19	-35	-869	-979	-55	17
3	4	-8	-58	-704	-44	176	31
4	3	-29	-61	-249	751	197	3
5	2	-44	-49	251	1000	50	-25
6	1	-53	-27	621	675	-125	-25
7	0	-56	0	756	0	-200	0
8	-1	-53	27	621	-675	-125	25
9	-2	-44	49	251	-1000	50	25
10	-3	-29	61	-249	-751	197	-3
11	-4	-8	58	-704	44	176	-31
12	-5	19	35	-869	979	-55	-17
13	-6	52	-13	-429	1144	-286	39
14	-7	91	-91	1001	-1001	143	-13
A	280	37128	39780	6466460	10581480	426360	8398

s	$P_1'(s)$	$P_2'(s)$	$P_3'(s)$	$P_4'(s)$	$P_5'(s)$	$P_6'(s)$	$P_7'(s)$
0	-1	-42	-568	-12425	-119881	-68306	-94762
1	-1	-36	-373	-5190	-20236	3592	19716
2	-1	-30	-208	-475	22499	17770	18022
3	-1	-24	-73	2140	29114	8048	-5519
4	-1	-18	32	3075	16619	-4954	-14682
5	-1	-12	107	2750	-1756	-11176	-6772
6	-1	-6	152	1585	-16561	-9478	6486
7	-1	0	167	0	-22126	0	12597
8	-1	6	152	-1585	-16561	9478	6486
9	-1	12	107	-2750	-1756	11176	-6772
10	-1	18	32	-3075	16619	4954	-14682
11	-1	24	-73	-2140	29114	-8048	-5519
12	-1	30	-208	475	22499	-17770	18022
13	-1	36	-373	5190	-20236	-3592	19716
14	-1	42	-568	12425	-119881	68306	-94762
D	1h	1h	6h	6h	30h	60h	420h

Tables. Orthogonal Polynomials and Numerical Differentiation Tables for n + 1 Equally Spaced Points

n = 15(16 points)

s	$P_1(s)$	$P_2(s)$	$P_3(s)$	$P_4(s)$	$P_5(s)$	$P_6(s)$	$P_7(s)$
0	15	35	455	273	143	65	195
1	13	21	91	-91	-143	-117	-533
2	11	9	-143	-221	-143	-39	143
3	9	-1	-267	-201	-33	59	423
4	7	-9	-301	-101	77	87	157
5	5	-15	-265	23	131	45	-235
6	3	-19	-179	129	115	-25	-375
7	1	-21	-63	189	45	-75	-175
8	-1	-21	63	189	-45	-75	175
9	-3	-19	179	129	-115	-25	375
10	-5	-15	265	23	-131	45	235
11	-7	-9	301	-101	-77	87	-157
12	-9	-1	267	-201	33	59	-423
13	-11	9	143	-221	143	-39	-143
14	-13	21	-91	-91	143	-117	533
15	-15	35	-455	273	-143	65	-195
A	1360	5712	1007760	470288	201552	77520	1545232

s	$P_1'(s)$	$P_2'(s)$	$P_3'(s)$	$P_4'(s)$	$P_5'(s)$	$P_6'(s)$	$P_7'(s)$
0	-2	-15	-1307	-3075	-15197	-4435	-577785
1	-2	-13	-887	-1391	-3227	39	101551
2	-2	-11	-527	-253	2353	1133	125099
3	-2	-9	-227	423	3703	687	-9021
4	-2	-7	13	721	2623	-119	-85909
5	-2	-5	193	725	553	-645	-64695
6	-2	-3	313	519	-1427	-671	8871
7	-2	-1	373	187	-2597	-277	67839
8	-2	1	373	-187	-2597	277	67839
9	-2	3	313	-519	-1427	671	8871
10	-2	5	193	-725	553	645	-64695
11	-2	7	13	-721	2623	119	-85909
12	-2	9	-227	-423	3703	-687	-9021
13	-2	11	-527	253	2353	-1133	125099
14	-2	13	-887	1391	-3227	-39	101551
15	-2	15	-1307	3075	-15197	4435	-577785
D	1h	1h	3h	6h	30h	10h	210h

Tables. Orthogonal Polynomials and Numerical Differentiation Tables for n + 1 Equally Spaced Points

n = 16(17 points)

s	$P_1(s)$	$P_2(s)$	$P_3(s)$	$P_4(s)$	$P_5(s)$	$P_6(s)$	$P_7(s)$
0	8	40	28	52	104	104	130
1	7	25	7	-13	-91	-169	-325
2	6	12	-7	-39	-104	-78	39
3	5	1	-15	-39	-39	65	247
4	4	-8	-18	-24	36	128	149
5	3	-15	-17	-3	83	93	-75
6	2	-20	-13	17	88	2	-215
7	1	-23	-7	31	55	-85	-175
8	0	-24	0	36	0	-120	0
9	-1	-23	7	31	-55	-85	175
10	-2	-20	13	17	-88	2	215
11	-3	-15	17	-3	-83	93	75
12	-4	-8	18	-24	-36	128	-149
13	-5	1	15	-39	39	65	-247
14	-6	12	7	-39	104	-78	-39
15	-7	25	-7	-13	91	-169	325
16	-8	40	-28	52	-104	104	-130
A	408	7752	3876	16796	100776	178296	579462

s	$P_1'(s)$	$P_2'(s)$	$P_3'(s)$	$P_4'(s)$	$P_5'(s)$	$P_6'(s)$	$P_7'(s)$
0	-1	-16	-149	-536	-9926	-6192	-214595
1	-1	-14	-104	-259	-2501	-203	30510
2	-1	-12	-65	-66	1204	1516	50348
3	-1	-10	-32	55	2359	1135	5534
4	-1	-8	-5	116	1954	104	-28087
5	-1	-6	16	129	799	-727	-29410
6	-1	-4	31	106	-476	-988	-7500
7	-1	-2	40	59	-1421	-669	17518
8	-1	0	43	0	-1766	0	28109
9	-1	2	40	-59	-1421	669	17518
10	-1	4	31	-106	-476	988	-7500
11	-1	6	16	-129	799	727	-29410
12	-1	8	-5	-116	1954	-104	-28087
13	-1	10	-32	-55	2359	-1135	5534
14	-1	12	-65	66	1204	-1516	50348
15	-1	14	-104	259	-2501	203	30510
16	-1	16	-149	536	-9926	6192	-214595
D	1h	1h	6h	6h	30h	10h	140h

Tables. Orthogonal Polynomials and Numerical Differentiation Tables for n + 1 Equally Spaced Points

n = 17(18 points)

s	$P_1(s)$	$P_2(s)$	$P_3(s)$	$P_4(s)$	$P_5(s)$	$P_6(s)$	$P_7(s)$
0	17	68	68	68	884	442	442
1	15	44	20	-12	-676	-650	-1014
2	13	23	-13	-47	-871	-377	-13
3	11	5	-33	-51	-429	169	715
4	9	-10	-42	-36	156	481	585
5	7	-22	-42	-12	588	439	-31
6	5	-31	-35	13	733	145	-563
7	3	-37	-23	33	583	-209	-651
8	1	-40	-8	44	220	-440	-280
9	-1	-40	8	44	-220	-440	280
10	-3	-37	23	33	-583	-209	651
11	-5	-31	35	13	-733	145	563
12	-7	-22	42	-12	-588	439	31
13	-9	-10	42	-36	-156	481	-585
14	-11	5	33	-51	429	169	-715
15	-13	23	13	-47	871	-377	13
16	-15	44	-20	-12	676	-650	1014
17	-17	68	-68	68	-884	442	-442
A	1938	23256	23256	28424	6953544	2941884	5794620

s	$P_1'(s)$	$P_2'(s)$	$P_3'(s)$	$P_4'(s)$	$P_5'(s)$	$P_6'(s)$	$P_7'(s)$
0	-2	-51	-337	-323	-38263	-46597	-623139
1	-2	-45	-241	-165	-11023	-3355	67453
2	-2	-39	-157	-52	3362	10712	153056
3	-2	-33	-85	22	8672	9504	39880
4	-2	-27	-25	63	8147	2631	-65855
5	-2	-21	23	77	4487	-3927	-92399
6	-2	-15	59	70	-148	-7160	-47872
7	-2	-9	83	48	-4138	-6368	23416
8	-2	-3	95	17	-6403	-2501	73725
9	-2	3	95	-17	-6403	2501	73725
10	-2	9	83	-48	-4138	6368	23416
11	-2	15	59	-70	-148	7160	-47872
12	-2	21	23	-77	4487	3927	-92399
13	-2	27	-25	-63	8147	-2631	-65855
14	-2	33	-85	-22	8672	-9504	39880
15	-2	39	-157	52	3362	-10712	153056
16	-2	45	-241	165	-11023	3355	67453
17	-2	51	-337	323	-38263	46597	-623139
D	1h	2h	6h	3h	15h	20h	140h

Tables. Orthogonal Polynomials and Numerical Differentiation Tables for n + 1 Equally Spaced Points

$n = 18$ (19 points)

s	$P_1(s)$	$P_2(s)$	$P_3(s)$	$P_4(s)$	$P_5(s)$	$P_6(s)$	$P_7(s)$
0	9	51	204	612	102	1326	306
1	8	34	68	-68	-68	-1768	-646
2	7	19	-28	-388	-98	-1222	-96
3	6	6	-89	-453	-58	234	411
4	5	-5	-120	-354	3	1235	425
5	4	-14	-126	-163	54	1352	97
6	3	-21	-112	42	79	729	-267
7	2	-26	-83	227	74	-214	-427
8	1	-29	-44	352	44	-1012	-308
9	0	-30	0	396	0	-1320	0
10	-1	-29	44	352	-44	-1012	308
11	-2	-26	83	227	-74	-214	427
12	-3	-21	112	42	-79	729	267
13	-4	-14	126	-168	-54	1352	-97
14	-5	-5	120	-354	-3	1235	-425
15	-6	6	89	-453	58	234	-411
16	-7	19	28	-388	98	-1222	86
17	-8	34	-68	-68	68	-1768	646
18	-9	51	-204	612	-102	1326	-306
A	570	13566	213180	2289132	89148	24515700	2451570

s	$P'_1(s)$	$P'_2(s)$	$P'_3(s)$	$P'_4(s)$	$P'_5(s)$	$P'_6(s)$	$P'_7(s)$
0	-1	-18	-946	-5391	-16151	-125241	-374957
1	-1	-16	-691	-2889	-5186	-13632	27977
2	-1	-14	-466	-1057	889	26537	93987
3	-1	-12	-271	186	3424	27296	36748
4	-1	-10	-106	925	3589	11435	-28625
5	-1	-8	29	1244	2374	-6176	-55547
6	-1	-6	134	1227	589	-17267	-40553
7	-1	-4	209	958	-1136	-18848	-2508
8	-1	-2	254	521	-2351	-11889	33563
9	-1	0	269	0	-2786	0	48025
10	-1	2	254	-521	-2351	11889	33563
11	-1	4	209	-958	-1136	18848	-2508
12	-1	6	134	-1227	559	17267	-40553
13	-1	8	29	-1244	2374	6176	-55547
14	-1	10	-106	-925	3589	-11435	-28625
15	-1	12	-271	-186	3424	-27296	36748
16	-1	14	-466	1057	889	-26537	93987
17	-1	16	-691	2888	-5186	13632	27977
18	-1	18	-946	5391	-16151	125241	-374957
D	1h	1h	6h	6h	60h	20h	140h

Tables. Orthogonal Polynomials and Numerical Differentiation Tables for n + 1 Equally Spaced Points

n = 19(20 points)

s	$P_1(s)$	$P_2(s)$	$P_3(s)$	$P_4(s)$	$P_5(s)$	$P_6(s)$	$P_7(s)$
0	19	57	969	1938	1938	1938	646
1	17	39	357	-102	-1122	-2346	-1258
2	15	23	-85	-1122	-1802	-1870	-306
3	13	9	-377	-1402	-1222	6	702
4	11	-3	-539	-1187	-187	1497	891
5	9	-13	-591	-687	771	1931	387
6	7	-21	-553	-77	1351	1353	-321
7	5	-27	-445	503	1441	195	-777
8	3	-31	-287	948	1076	-988	-756
9	1	-33	-99	1188	396	-1716	-308
10	-1	-33	99	1188	-396	-1716	308
11	-3	-31	287	948	-1076	-988	756
12	-5	-27	445	503	-1441	195	777
13	-7	-21	553	-77	-1351	1353	321
14	-9	-13	591	-687	-771	1931	-387
15	-11	-3	539	-1187	187	1497	-891
16	-13	9	377	-1402	1222	6	-702
17	-15	23	85	-1122	1802	-1870	306
18	-17	39	-357	-102	1122	-2346	1258
19	-19	57	-969	1938	-1938	1938	-646
A	2660	17556	4903140	22881320	31201800	49031400	9806280

s	$P'_1(s)$	$P'_2(s)$	$P'_3(s)$	$P'_4(s)$	$P'_5(s)$	$P'_6(s)$	$P'_7(s)$
0	-2	-19	-2111	-31825	-282629	-165623	-2093841
1	-2	-17	-1571	-17765	-99299	-23749	87303
2	-2	-15	-1091	-7275	6541	31765	525111
3	-2	-13	-671	65	55051	37479	266783
4	-2	-11	-311	4675	63871	20053	-94781
5	-2	-9	-11	6975	48121	-2433	-291621
6	-2	-7	229	7385	20401	-19159	-269837
7	-2	-5	409	6325	-9209	-25245	-97189
8	-2	-3	529	4215	-33149	-20431	110823
9	-2	-1	589	1475	-46379	-7757	246399
10	-2	1	589	-1475	-46379	7757	246399
11	-2	3	529	-4215	-33149	20431	110823
12	-2	5	409	-6325	-9209	25245	-97189
13	-2	7	229	-7385	20401	[illegible]	-269837
14	-2	9	-11	-6975	48121	[illegible]	-291621
15	-2	11	-311	-4675	63871	-20053	-94781
16	-2	13	-671	-65	55051	-37479	266783
17	-2	15	-1091	7275	6541	-31765	525111
18	-2	17	-1571	17765	-99299	23749	87303
19	-2	19	-2111	31825	-282629	165623	-2093841
D	1h	1h	3h	12h	60h	20h	420h

Tables. Orthogonal Polynomials and Numerical Differentiation Tables for n + 1 Equally Spaced Points

n = 20(21 points)

s	$P_1(s)$	$P_2(s)$	$P_3(s)$	$P_4(s)$	$P_5(s)$	$P_6(s)$	$P_7(s)$
0	10	190	570	969	3876	6460	3230
1	9	133	228	0	-1938	-7106	-5814
2	8	82	-24	-510	-3468	-6392	-2006
3	7	37	-196	-680	-2618	-918	2754
4	6	-2	-298	-615	-788	3996	4266
5	5	-35	-340	-406	1063	6075	2565
6	4	-62	-332	-130	2354	5088	-543
7	3	-83	-284	150	2819	2001	-3087
8	2	-98	-206	385	2444	-1716	-3822
9	1	-107	-108	540	1404	-4628	-2548
10	0	-110	0	594	0	-5720	0
11	-1	-107	108	540	-1404	-4628	2548
12	-2	-98	206	385	-2444	-1716	3822
13	-3	-83	284	150	-2819	2001	3087
14	-4	-62	332	-130	-2354	5088	543
15	-5	-35	340	-406	-1063	6075	-2565
16	-6	-2	298	-615	788	3996	-4266
17	-7	37	196	-680	2618	-918	-2754
18	-8	82	24	-510	3468	-6392	2006
19	-9	133	-228	0	1938	-7106	5814
20	-10	190	-570	969	-3876	6460	-3230
A	770	201894	1730520	5720330	121687020	514829700	223092870

s	$P'_1(s)$	$P'_2(s)$	$P'_3(s)$	$P'_4(s)$	$P'_5(s)$	$P'_6(s)$	$P'_7(s)$
0	-1	-60	-1171	-7459	-523756	-1511160	-9341610
1	-1	-54	-886	-4311	-198571	-265587	91149
2	-1	-48	-631	-1928	-4056	257376	2306691
3	-1	-42	-406	-217	92069	347459	1421821
4	-1	-36	-211	906	118004	220472	-132186
5	-1	-30	-46	1525	98369	27545	-1159135
6	-1	-24	89	1724	53954	-135632	-1307521
7	-1	-18	194	1587	1769	-219569	-740199
8	-1	-12	269	1198	-44956	-211736	135886
9	-1	-6	314	641	-76771	-127323	879909
10	-1	0	329	0	-88036	0	1167315
11	-1	6	314	-641	-76771	127323	879909
12	-1	12	269	-1198	-44956	211736	135886
13	-1	18	194	-1587	1769	219569	-740199
14	-1	24	89	-1724	53954	135632	-1307521
15	-1	30	-46	-1525	98369	-27545	-1159135
16	-1	36	-211	-906	118004	-220472	-132186
17	-1	42	-406	217	92069	-347459	1421821
18	-1	48	-631	1928	-4056	-257376	2306691
19	-1	54	-886	4311	-198571	265587	91149
20	-1	60	-1171	7459	-523756	1511160	-9341610
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13. ABSTRACT

The present method of reducing data by formulating a polynomial, taking the derivative and substituting experimental values, has been simplified for instances where the independent variable is evenly spaced and the derivative is desired at these given points only.

Orthogonal polynomials, resembling Legendre polynomials, are differentiated and tabulated in a useful form. When used in conjunction with available orthogonal polynomial tables, these new tables give derivatives more readily than the usual procedure. These tables present coefficients for obtaining first derivatives at 6 to 21 evenly spaced points. This report extends the available orthogonal polynomial tables to sixth and seventh degrees, when appropriate, and gives a demonstration of the method.

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